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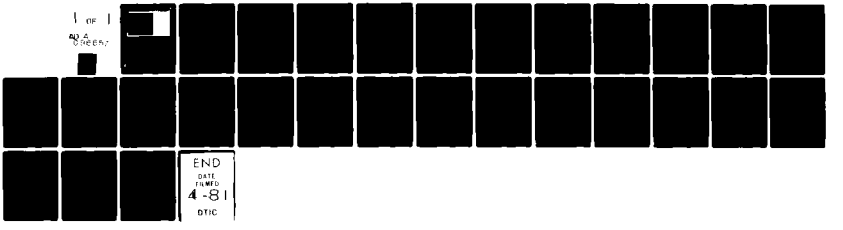

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THE SINGULARLY PERTURBED TURNING POINT PROBLEM: A SPECTRAL APPROACH

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THE SINGULARLY PERTURBED TURNING POINT PROBLEM: A SPECTRAL APPROACH

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ABSTRACT

We study the singularly perturbed two-point boundary value problem on the interval $(-1,1)$

$$-\varepsilon u'' + p(x)u' + q(x)u = 0, \quad u(-1) = A, \quad u(1) = B ,$$

in which the coefficient p has zeros in the interval $(-1,1)$. For the case where p has precisely one zero we give a survey of our method of construction of an asymptotic approximation of the solution via the eigenfunction expansion, and we show that "Ackerberg - O'Malley resonance" is identical to ordinary resonance, namely that a free mode in the solution is amplified strongly by a small divisor. For the case where p has several zeros and where q is identically zero we construct an asymptotic approximation and prove its validity by a variational (Galerkin) method. The methods described in this paper can be generalized to turning point problems in several dimensions.

AMS (MOS) Subject Classifications: 34B05, 65L10

Key Words: Singular perturbations, Turning points, Resonance, Variational approximations

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

Consider the two-point boundary value problem

$$-\varepsilon u'' + xu' - \lambda u = 0, \quad u(-1) = A, \quad u(1) = B,$$

with $0 < \varepsilon << 1$. For certain values of λ, namely the eigenvalues, the problem need not have a solution; for $\varepsilon \to +0$ those eigenvalues tend to the non-negative integers. Although for values near such an eigenvalue the solution exists, it is very sensitive to small changes in λ and for a long time it has been unknown how to construct a rigorous asymptotic approximation to it. Yet an asymptotic approximation is of interest since the equation and its generalizations offer a deterministic model for the motion of a particle in a potential field, which executes a random walk under influence of small random forces. In this paper we construct asymptotic approximations to solutions of such problems, in which the coefficient of u' has one or several zeros (turning points) and we prove their validity.

THE SINGULARLY PERTURBED TURNING POINT PROBLEM: A SPECTRAL APPROACH

Pieter de Groen*

1. INTRODUCTION.

In a survey paper [1] on "the capriciousness of singular perturbations", Wasow gives a number of examples of singular perturbation problems which display unexpected behaviour. One of his examples is the following simple looking two-point boundary value problem on the interval $(-1,1)$

$$\varepsilon u'' - p(x)u' = f(x),\ u(-1) = u(1) = 0 \qquad (1.1)$$

in which the coefficient p changes sign at $x = 0$, $p(0) = 0$ and $p(x) \neq 0$ if $x \neq 0$. It looks natural that the solution converges to a solution of the reduced equation $pu' = f$ with the possible exception of neighbourhoods of the boundary and of the point $x = 0$, at which the reduced equation has a singularity. Referring to an unpublished part of his Ph.D.-thesis (1942) Wasow states:

(i) if $p'(0) < 0$ the solution of (1) converges on $[-1,0)$ and on $(0,1]$ to solutions of the reduced equation $pu' = f$, which satisfy the boundary conditions at $+1$ and -1 respectively; the limit is discontinuous at $x = 0$.

(ii) if $p'(0) > 0$ the solution of (1) diverges in general.

This divergence in (ii) is not "generic"; it is rather exceptional and is due to the lack of a zeroth order term in (1.1).

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Problems of type (1.1) have become famous by a paper of Ackerberg and O'Malley, in which they construct formal approximations to the solution of the problem

$$\varepsilon u'' - p(x)u' + q(x)u = 0,\ u(-1) = A, u(1) = B\ . \qquad (1.2)$$

Their approximations do not decay exponentially if $p'(0) > 0$ and $q(0)/p'(0)$ is a non-negative integer, a phenomenon they have called "resonance". This paper has been followed by a large number of other papers, studying this subject, mostly by formal methods.

The existence of "resonance" for problems of type (1.2) can be proved in several ways, all of which have in common the introduction of an additional parameter, which performs a "resolution of the singularity". Olver [3] constructs an approximation by linking together uniform approximations of independent solutions of the equation containing an additional parameter δ. The linking across the turning point yields conditions on $\delta(\varepsilon)$ for "resonance" to occur. Kopell [4] (see also this volume) introduces an additional parameter δ and studies the continuity with respect to δ of certain integral manifolds of solutions, whose geometric properties yield criteria for "resonance". Both approaches have in common, that they do not construct an approximation of the solution of (1.2), but that they determine a function $\delta(\varepsilon)$ such that the perturbed problem

$$\varepsilon u'' - p(x)u' + q(x)u + \delta(\varepsilon)u = 0,\ u(-1) = A, u(1) = B\ , \qquad (1.3)$$

displays "resonance". In our approach [5], before we try to solve (1.2), we first analyse the associated homogeneous problem

$$\varepsilon u'' - p(x)u' + q(x)u + \lambda u = 0,\ u(-1) = u(1) = 0 \qquad (1.4)$$

which can be recognized as the eigenvalue problem associated with (1.2). Formally this looks not much different from (1.3), yet the spectral analysis yields a quite different description of approximations of the solution of (1.2).

In this contribution we shall give an overview of the paper [5] and we shall give for the particular problem (1.1)

with several turning points an analysis, which can be generalized to the analogous problem in several dimensions. This particular problem (1.1) has applications to exit problems for Brownian motion in potential wells, cf. [6].

2. MOTIVATION BY AN EXAMPLE.

In order to motivate the spectral approach, we consider the particular problem

$$\varepsilon u'' - xu' + \lambda u = 0,\ u(-1) = A,\ u(1) = B\ . \tag{2.1}$$

Its exact solution u_ε can be expressed in terms of confluent hypergeometric functions,

$$u_\varepsilon(x) = \frac{1}{2}(A + B)\frac{F(-\frac{1}{2}\lambda;\frac{1}{2};x^2/2\varepsilon)}{F(-\frac{1}{2}\lambda;\frac{1}{2},1/2\varepsilon)} \tag{2.2}$$

$$+ \frac{1}{2}(B - A)\frac{xF(\frac{1}{2} - \frac{1}{2}\lambda;3/2;x^2/2\varepsilon)}{F(\frac{1}{2} - \frac{1}{2}\lambda;3/2;1/2\varepsilon)}\ ,$$

provided the denominators are non-zero. Since they are non-zero for each fixed λ and ε small enough (non-uniformly depending on λ), we find the asymptotic formulae for $\varepsilon \to 0$,

$$u_\varepsilon(x) \sim A\exp\{(-1 - x)/\varepsilon\} + B\exp\{(x - 1)/\varepsilon\}, \tag{2.3a}$$

$$\text{if}\quad \lambda \neq 0,1,2,\ldots,$$

$$u_\varepsilon(x) \sim \frac{1}{2}(B + (-1)^\lambda A)x^\lambda \tag{2.3b}$$

$$+ \frac{1}{2}(B - (-1)^\lambda A)\exp\{-(x - 1)^2/2\varepsilon\},$$

$$\text{if}\quad \lambda = 0,1,2,\ldots,$$

valid for constant λ.

If one of the denominators in (2.2) is zero for some $\lambda(\varepsilon)$, a solution of (2.1) need not exist, the homogeneous problem

$$\varepsilon u'' - xu' + \lambda(\varepsilon)u = 0,\ u(-1) = u(1) = 0\ , \tag{2.4}$$

has a non-trivial solution (eigenfunction) and $\lambda(\varepsilon)$ is an eigenvalue. Actually, it is well-known that the denominators in (2.2) as functions of λ have denumerably many zeros $\lambda_k(\varepsilon)$ for each $\varepsilon > 0$; Sturm-Liouville theory implies that the corresponding set of eigenfunctions $e_k(x,\varepsilon)$ is complete.

Let us now assume that Z_ε is a formal approximation to the solution of (2.1), satisfying the boundary conditions

$Z_\varepsilon(-1) = A$ and $Z_\varepsilon(1) = B$ exactly, then we can insert it in equation (2.1) and expand its residue in the eigenfunctions,

$$\varepsilon Z_\varepsilon'' - xZ_\varepsilon' + \lambda Z_\varepsilon = \sum_k \alpha_k e_k \ .$$

Sturm-Liouville theory then implies

$$u_\varepsilon - Z_\varepsilon = \sum_k \frac{\alpha_k e_k}{\lambda - \lambda_k(\varepsilon)} \ . \tag{2.5}$$

We see from this expression that Z_ε can be a good approximation, unless one of the denominators in the right-hand side is small. If a denominator in (2.5) is small enough, it can amplify the corresponding eigenfunction in the expansion of $u_\varepsilon - Z_\varepsilon$ so strongly, that it is the dominating term in the approximation of u_ε. Clearly, "resonance" in the sense of Ackerberg & O'Malley or in the sense of Kopell is identical to the well-known phenomenon of resonance in Mechanics, namely, that a free mode is strongly amplified by a small divisor.

Since equation (2.4) is transformed to Hermite's equation by the change of the independent variable $x = \varepsilon^{\frac{1}{2}}\xi$ on a ξ-interval which becomes unbounded for $\varepsilon \to 0$, the eigenvalues of (2.4) tend to the non-negative integers. This fact is reflected in the asymptotic formula (2.3b).

This example indicates, that, before trying to approximate the solution of the general problem (1.2), we should analyze the spectrum of the associated differential operator L_ε,

$$L_\varepsilon u := -\varepsilon u'' + pu' - qu \tag{2.6}$$

and that we should show existence of a solution of (1.2). It is this aspect that distinguishes our approach from all other ones.

3. FIRST-ORDER APPROXIMATIONS OF THE EIGENVALUES.

For a study of eigenvalues and for application of Sturm-Liouville theory the operator L_ε, defined in (2.6), does not look very suitable, since it is not selfadjoint in the usual space of square integrable functions. This can be amended in two equivalent ways:

(i) Perform the Liouville transformation,

$$v(x) = u(x)J_\varepsilon(x),\ J_\varepsilon(x) := \exp\{-\frac{1}{2\varepsilon}\int_0^x p(t)dt\}\ . \quad (3.1)$$

It transforms the operator (2.6) and the eigenvalue equation (1.4) to a selfadjoint form,

$$M_\varepsilon v := -\varepsilon v'' + (\frac{p^2}{4\varepsilon} - \frac{1}{2}p' - q)v = \lambda v,\ v(0) = v(1) = 0. \quad (3.2)$$

(ii) Consider the operator L_ε in the space of weighted square integrable functions, in which the inner product is given by

$$(u,v)_w := \int_{-1}^{1} u(x)v(x)J_\varepsilon^2(x)dx\ . \quad (3.3)$$

With respect to this inner product L_ε is selfadjoint.

Both methods have their virtues. We shall employ the first one in order to derive a first approximation to the eigenvalues. Once the convergence of the eigenvalues to well-separated limits is established, the second method is easier for setting up an iterative scheme by which we can obtain approximations of higher order.

The tool we use for the derivation of a first estimate is Rayleigh's minimax characterization of the eigenvalues of a selfadjoint operator. Let

$$\sigma(M_\varepsilon) = \{\lambda_k(\varepsilon)\,|\,k = 0,1,\ldots\} \quad \text{with} \quad \lambda_k < \lambda_{k+1} \quad (3.4)$$

be the set of eigenvalues of M_ε, ordered in increasing sense, then the k-th eigenvalue satisfies, cf. [7],

$$\lambda_k(\varepsilon) = \min_{E\subset H_0^1(-1,1),\dim E=k+1}\ \max_{u\in E,\,\|u\|=1} (M_\varepsilon u,u)\ , \quad (3.5)$$

where $(\cdot,\cdot)$ is the usual inner product in L^2 and $\|\cdot\|$ the associated norm.

From (3.5) it is clear that each set of $k+1$ independent trial functions yields an upper bound for the k-th eigenvalue. If $p(0)$ is the only zero of p in $[-1,1]$,

we find a suitable set of trial functions as follows. Apply to equation (3.2) the stretching $\xi = x\sqrt{|p'(0)|/2\varepsilon}$ and expand the coefficients in powers of $\sqrt{\varepsilon}$, then we find the lowest order part

$$-\frac{d^2v}{d\xi^2} + \xi^2 v = \left\{\frac{p'(0)}{2|p'(0)|} + \frac{\lambda + q(0)}{|p'(0)|}\right\} v \ . \tag{3.6}$$

This equation has a decaying solution only if the coefficient of v in the right-hand side is a positive half-odd integer, say $j + \frac{1}{2}$, namely $H_j(\xi)\exp(-\frac{1}{2}\xi^2)$. Using the first $k+1$ of those as trial functions for λ_k (correcting them such that they are zero at the boundary) we obtain the upper bound

$$\lambda_k(\varepsilon) \leq \begin{cases} k\,p'(0) - q(0) + C_k\varepsilon^{\frac{1}{2}} & \text{if } p'(0) > 0\,, \\ (k+1)p'(0) - q(0) + C_k\varepsilon^{\frac{1}{2}} & \text{if } p'(0) < 0\,. \end{cases} \tag{3.7}$$

The minimax in (3.5) decreases if the minimizing subspace E is taken from the larger space of functions which need not be continuous at $x = \pm\varepsilon^{\frac{1}{2}}$ and which have square integrable derivatives only in the subintervals $(-1,-\varepsilon^{\frac{1}{2}})$, $(-\varepsilon^{\frac{1}{2}},+\varepsilon^{\frac{1}{2}})$ and $(\varepsilon^{\frac{1}{2}},1)$. Hence λ_k is larger than the k-th eigenvalue in the joint spectrum of the restrictions of M_ε to those subintervals with Neumann boundary conditions. Since the potential term in M_ε on the subintervals $(-1,-\varepsilon^{\frac{1}{2}})$ and $(\varepsilon^{\frac{1}{2}},1)$ is bounded from below by $\gamma\varepsilon^{-\frac{1}{2}}$ for some $\gamma > 0$, the smallest eigenvalue of these restrictions is larger than $\gamma\varepsilon^{-\frac{1}{2}}$. On $(-\varepsilon^{\frac{1}{2}},\varepsilon^{\frac{1}{2}})$ the potential of M_ε is estimated from below by V_ε,

$$V_\varepsilon(x) := \frac{x^2}{4\varepsilon}\,(|p'(0)| - \alpha\varepsilon^{\frac{1}{2}})^2 - \frac{1}{2}\,p'(0) - q(0) - \beta\varepsilon^{\frac{1}{2}}\,, \tag{3.8}$$

if α and β are chosen suitably. Hence λ_k is bounded from below by the k-th eigenvalue $\pi_k(\varepsilon)$ of the problem

$$-\varepsilon u'' + V_\varepsilon(x)u = \lambda u,\ u'(\pm\varepsilon^{\frac{1}{2}}) = 0\ . \tag{3.9}$$

For this problem it is (with the stretching $\xi = x\sqrt{\varepsilon}$) not difficult to show, that its eigenvalues satisfy

$$\pi_k(\varepsilon) = \begin{cases} kp'(0) - q(0) + 0(\varepsilon^{\frac{1}{2}}) & \text{if } p'(0) > 0, \\ (k+1)p'(0) - q(0) + 0(\varepsilon^{\frac{1}{2}}) & \text{if } p'(0) < 0. \end{cases} \tag{3.10}$$

So we have given a sketch of the proof of the following theorem:

Theorem 1: The eigenvalues of L_ε satisfy for $\varepsilon \to 0$ the estimates:

$$\lambda_k(\varepsilon) = \begin{cases} kp'(0) - q(0) + 0(\varepsilon^{\frac{1}{2}}) & \text{if } p'(0) > 0, \\ (k+1)p'(0) - q(0) + 0(\varepsilon^{\frac{1}{2}}) & \text{if } p'(0) < 0. \end{cases} \tag{3.11}$$

For details of the proof we refer to [5].

If p has several simple zeros in the interior of the interval, every zero produces a set of eigenvalues, which satisfies the analogue of (3.11). If p has a zero at the boundary, this zero produces "half" of the above set of eigenvalues, namely only the ones with odd index, the even ones being ruled out by the boundary condition. The proof is analogous to the one above. For every zero of p we now construct a set of Hermite-functions as trial functions; this yields an upper bound. For a lower bound we make the same type of restrictions to $0(\varepsilon^{\frac{1}{2}})$-neighbourhoods of the zeros of p as above.

4. APPROXIMATIONS OF HIGHER ORDER TO THE EIGENVALUES AND EIGENFUNCTIONS.

In order to obtain approximations of higher order, we return to the original operator L_ε, which is selfadjoint with respect to the weighted inner product (3.3). We assume now that $p'(0)$ is positive, such that the weight function J_ε^2 takes its maximum at $x = 0$ and is exponentially small (relative to the maximum) outside a small neighbourhood of this point. This implies that our norm hardly notices errors of an approximation if they are only of polynomial order in ε. Therefore the algorithm we devise here for the approximation of the eigenfunction does not care for errors outside a neighbourhood of the point $x = 0$.

For the construction of a formal approximation of the k-th eigenvalue $\lambda_k(\varepsilon)$ and the associated eigenfunction

$e_k(x,\varepsilon)$ we perform the stretching $\xi := x\sqrt{p'(0)/2\varepsilon}$ in the eigenvalue equation (1.4), we insert the formal expansions

$$2p(\xi\sqrt{2\varepsilon/p'(0)})/\sqrt{2\varepsilon p'(0)} = 2\xi + \sum_{i=1}^{\infty} p_i \varepsilon^{\frac{1}{2}i} \xi^{i+1} \tag{4.1}$$

$$2q(\xi\sqrt{2\varepsilon/p'(0)})/p'(0) = \sum_{i=0}^{\infty} q_i \varepsilon^{\frac{1}{2}i} \xi^{i} \tag{4.2}$$

$$\lambda_k(\varepsilon) = kp'(0) - q(0) + \frac{1}{2} p'(0) \sum_{j=1}^{\infty} \lambda_{kj} \varepsilon^{\frac{1}{2}j} \tag{4.3}$$

$$e_k(\xi\sqrt{2\varepsilon/p'(0)},\varepsilon) = \sum_{j=0}^{\infty} e_{kj}(\xi) \varepsilon^{\frac{1}{2}j} \tag{4.4}$$

and we collect equal powers of ε. This results in the recursive system of equations ($e' = de/d\xi$)

$$e''_{kj} - 2\xi e'_{kj} + 2k e_{kj} \tag{4.5}$$

$$= \sum_{i=1}^{j} (p_i \xi^{i+1} \frac{d}{d\xi} + q_i \xi^i + \lambda_{ki}) e_{k,j-i-1} ,$$

from which the unknown functions e_{kj} and the coefficients λ_{ki} are determined. Clearly, e_{k0} satisfies Hermite's equation; all its solutions are exponentially growing except the Hermite polynomial H_k. Hence we may choose $e_{k0} = H_k$. The next term e_{k1} satisfies Hermite's equation with a polynomial right-hand side,

$$e''_{k1} - 2\xi e'_{k1} + 2k e_{k1} = p_1 \xi^2 H'_k + (q_1 \xi + \lambda_{k1}) H_k . \tag{4.6}$$

Again its solution is exponentially increasing, unless it is a polynomial. The particular equation

$$y'' - 2\xi y' + 2ky = H_j \tag{4.7}$$

has the polynomial solution $y = H_j/(2k - 2j)$, provided $k \neq j$. If $k = j$ every solution of it is exponentially increasing. Writing the right-hand side as a sum of Hermite polynomials, we see that a unique coefficient λ_{k1} exists, such that this right-hand side does not contain a multiple of H_k. More generally, in every step of the recursion (4.5) we

can determine a unique λ_{kj} which suppresses the presence of exponentially growing terms in the solution. Finally we show that the odd coefficients $\lambda_{k,2i-1}$ $(i = 1,2...)$ must be zero. If k is even (odd), then H_k is an even (odd) function and the right-hand side of (4.6) is odd (even) provided $\lambda_{k1} = 0$ and cannot contain a non-zero multiple of H_k. Analogously, if j is odd and $\lambda_{ki} = 0$ $(i = 1,3,...,j)$, then the right-hand side of (4.5) is odd (even) is k is even (odd).

In order to prove validity of those expansions, we define the partial sums Λ_{kj} and E_{kj} by

$$\Lambda_{kj}(\varepsilon) := kp'(0) - q(0) + \frac{1}{2} p'(0) \sum_{i=1}^{j-1} \lambda_{k,2i} \varepsilon^i$$

$$E_{kj}(x,\varepsilon) := \rho(x) \sum_{i=0}^{2j-1} e_{ki}(x\sqrt{2\varepsilon/p'(0)})\varepsilon^{\frac{1}{2}i} , \tag{4.8}$$

where ρ is a C^∞ cut-off function,

$$\rho(x) = 1 \quad \text{if} \quad |x| < \frac{1}{2}, \quad \rho(x) = 0 \quad \text{if} \quad |x| > \frac{3}{4} . \tag{4.9}$$

The above construction implies

$$\|(L_\varepsilon - \Lambda_{kj})E_{kj}\|_w = 0(\varepsilon^j \|E_{kj}\|_w) . \tag{4.10}$$

Expanding E_{kj} in the true eigenfunctions and using the initial estimate (3.11) we easily find (cf. [5]):

Theorem 2: The eigenvalues of L_ε admit the asymptotic expansion

$$\lambda_k(\varepsilon) = kp'(0) - q(0) + \frac{1}{2}p'(0) \sum_{i=1}^{j-1} \lambda_{k,2i}\varepsilon^i + 0(\varepsilon^j) , \tag{4.11}$$

$$(\varepsilon \to 0), \ \forall j ,$$

and the associated eigenfunctions satisfy:

$$\|E_{kj} - \mu_{kj} e_k\|_w = 0(\varepsilon^j \|E_{kj}\|_w), \tag{4.12}$$

where $\mu_{kj} := \|e_k\|_w / \|E_{kj}\|_w$.

Obviously, the estimate (4.12) in the weighted norm can give good pointwise estimates of the error only in a

neighbourhood of diameter $0(\sqrt{\varepsilon})$ of the maximum of the weight function. Such a pointwise estimate can be obtained from the following variant of Sobolev's inequality,

$$u^2(x)J_\varepsilon^2(x) = \int_a^x \frac{d}{dx}\,(u^2(x)J_\varepsilon^2(x))dx = \tag{4.13}$$

$$= 2(u,u')_w - (pu,u)_w/\varepsilon \le C\|u\|_w^2/\varepsilon + \varepsilon\|u'\|_w^2 \le$$

$$\le C_1\|u\|_w^2/\varepsilon + \|u\|_w\|L_\varepsilon u\|_w,\ \forall u \in H_0^1(-1,1)$$

where we used the identity

$$\|u'\|_w^2 = (-u'' + pu'/\varepsilon,u)_w = (L_\varepsilon u + qu,u)_w/\varepsilon\ . \tag{4.14}$$

These estimates (4.12-13) imply the pointwise estimate

$$|E_{kj}(x,\varepsilon) - u_{kj}e_k(x,\varepsilon)| \le C_\gamma \varepsilon^{j-\frac{1}{2}},\ \forall x \in (-\gamma\sqrt{\varepsilon},\gamma\sqrt{\varepsilon})\ . \tag{4.15}$$

The approximations E_{kj} to the eigenfunction e_k we have constructed up to this point, are in fact the internal boundary layer terms, valid only in a neighbourhood of the turning point $x = 0$. In order to construct a uniformly valid approximation we have to match the internal layer terms to regular expansions valid in $(-1,0)$ and $(0,1)$. The integration constants are uniquely determined by the matching (cf. [5]). These regular expansions are matched to the boundary conditions in ordinary boundary layers. If we start with the normalization $e_{k0} = H_k$, the approximation is of order unity in an $0(\sqrt{\varepsilon})$-neighbourhood of $x = 0$ and hence of the order $0(\varepsilon^{-k/2})$ globally on the interval.

The validity of this formal approximation can be proved by common barrier function techniques. Since we already have a good approximation in an $0(\sqrt{\varepsilon})$-neighbourhood of $x = 0$, we can restrict the problem to subintervals $(-1,-\gamma\sqrt{\varepsilon})$ and $(\gamma\sqrt{\varepsilon},1)$ for a suitably chosen $\gamma > 0$, where the lowest order term of the regular expansion multiplied by $\log(x)$ is a good barrier function, cf. [5 & 7].

Remarks:

(i) If $p'(0) < 0$ we can prove the analogue of Theorem 2 by considering the L^2-adjoint L_ε^* of L_ε,

$$L_\varepsilon^* u := -\varepsilon u'' - (pu)' - qu, \quad u \in H_0^1 . \tag{4.16}$$

It has the same eigenvalues as L_ε has, and the sign of p is reversed, such that we can apply Theorem 2 directly. The eigenfunctions e_k^* of L_ε^* and e_k of L_ε satisfy the relation

$$e_k^* J_\varepsilon^{-2} = e_k \tag{4.17}$$

(ii) In general there is no reason why the asymptotic series for $\lambda_k(\varepsilon)$ should converge, hence it is impossible to determine approximations of it with exponentially small errors, unless p and q are analytic.

(iii) If p has several zeros, the above method for determining asymptotic power series for the eigenvalues works equally well. It is not yet clear how we can construct uniformly valid approximations to the eigenfunctions on the whole interval.

5. EXPONENTIAL DECAY AND RESONANCE (THE CASE $p'(0) > 0$).

If no eigenvalue of L_ε is zero, a solution for problem (1.2) exist. For the case $p'(0) > 0$ we shall construct an approximation following the suggestion of the example in Section 2.

For problems of this type boundary layers are generally expected at both end points of the interval, since the singular solution of $L_\varepsilon u = 0$ is increasing near $x = +1$ and decreasing at $x = -1$. Hence we can by usual methods construct a formal approximation Z_ε, which consists of boundary layer terms at $x = \pm 1$ only, which is exponentially small in the interior of the interval and satisfies for some k the uniform estimate

$$L_\varepsilon Z_\varepsilon = 0(\varepsilon^k) \qquad (\varepsilon \to 0) \tag{5.1}$$

and the boundary conditions

$$Z_\varepsilon(-1) = A, \qquad Z_\varepsilon(1) = B \ . \tag{5.2}$$

We easily find the lowest order term:

$$Z_\varepsilon(x) = A \exp(p(1)(x-1)/\varepsilon) + B \exp(p(-1)(-1-x)/\varepsilon) + O(\varepsilon) \ . \tag{5.3}$$

Following the suggestion of formula (2.5) we expand the residue in the true eigenfunctions,

$$L_\varepsilon Z_\varepsilon = \sum_{j=0}^{\infty} \beta_j e_j, \qquad \beta_j(\varepsilon) := \frac{(L_\varepsilon Z_\varepsilon, e_j)_w}{(e_j, e_j)_w} \ .$$

Hence, if $\lambda_k(\varepsilon) \to 0$ $(\varepsilon \to 0)$,

$$\| u_\varepsilon - Z_\varepsilon - \beta_k e_k/\lambda_k \|_w = O(\| L_\varepsilon Z_\varepsilon \|_w) \ . \tag{5.5}$$

Since the weight function in the weighted inner product is exponentially small outside a neighbourhood of $x = 0$, the coefficient β_k is exponentially small (in [5] we compute it with a relative error of order $O(\sqrt{\varepsilon})$). If the asymptotic series (4.3 & 8) contains non-zero terms, then obviously the eigenfunction component in (5.5) is exponentially small and its presence in the approximation is not noticed. On the other hand, if all terms of the asymptotic series for $\lambda_k(\varepsilon)$ are zero, and if the coefficients p and q are of class C^∞ only, we cannot determine by asymptotic expansions in powers of ε whether $\lambda_k(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, nor can we decide whether $\beta_k(\varepsilon)/\lambda_k(\varepsilon)$ converges to a definite limit (if the denominator is non-zero). Hence, for general C^∞ coefficients we are unable to determine, whether a solution exist for each small enough ε and whether it converges for $\varepsilon \to 0$, if $|\lambda_k(\varepsilon)| \leq \varepsilon^N$, $\forall N$.

If we have analytic coefficients p and q and if the asymptotic series for $\lambda_k(\varepsilon)$ vanishes, then we can apply uniform reduction theory and connection formulae. If the real interval $(-1,1)$ is contained in the smallest of the disks in complex plane, in which the power series expansions of p, x/p and q are convergent, then it can be shown,

cf. Sibuya [9], that the equation $L_\varepsilon u = 0$ has a solution which converges uniformly on $[-1,1]$ to a non-trivial solution of the reduced equation $pu' - qu = 0$; i.e. it can be shown that the equation $L_\varepsilon u = 0$ shows "resonance" in the sense of Kopell [4], in that case. From this particular solution plus a boundary layer approximation we can construct easily an approximation to the solution of (1.2). By adding boundary layers at both sides, we can also find a good estimate of the k-th eigenvalue, which is of the same order as β_k is. We remark that it is not known, how to find a satisfactory estimate of $\lambda_k(\varepsilon)$ if the condition about the radii of convergence is not satisfied.

6. A MULTIPLE TURNING POINT PROBLEM.

6.a) Position of the problem. We now return to the example (1.1) of Wasow; however, instead of an inhomogeneous equation we study the case with inhomogeneous boundary conditions,

$$L_\varepsilon u := -\varepsilon u'' + p(x)u' = 0,\ u(-1) = A,\ u(1) = B\ , \qquad (6.1)$$

in which the coefficient p has several zeros in the interior of the interval $(-1,1)$. This problem can easily be solved exactly:

$$u_\varepsilon(x) = A + (B - A)\Psi(x)/\Psi(1), \qquad (6.2)$$

$$\Psi(x) := \int_{-1}^{x} \exp\{\int_{-1}^{t} p(s)ds/\varepsilon\}dt$$

From this formula it is easily seen that the solution has transition layers at the absolute maxima of a primitive of p and is almost constant elsewhere.

Nevertheless, the study of this problem is interesting from the point of view of its analogue in several dimensions, where an exact solution does not exist. In [6] and [10] a variational method is described which yields a formal approximation, both in one and in several dimensions. In this section we shall sketch a proof of the validity of this variational method in one dimension, without using the information (6.2) we have about the exact solution. The proof carries over to several dimensions, but is more complicated there.

For a simplification of the presentation we shall make the following assumptions

(i) all zeros of p are simple, $p(-1) > 0$ and $p(1) < 0$,

(ii) p has $2n + 1$ zeros; the zeros at which p' is negative are denoted by β_i, $i = 0,\ldots,n$, and the zeros at which p' is positive are denoted by α_i, $i = 1,\ldots,n$; moreover we define $\alpha_0 := -1$ and $\alpha_{n+1} := 1$, such that we have

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 \ldots < \beta_n < \alpha_{n+1} . \tag{6.3}$$

(iii) $A = 0$ and $B = 1$.

There is nothing deep in those assumptions, they relieve us of having to write down several alternatives in most of the formulae. E.g. a boundary layer at the boundary points ± 1 in general has a different structure than an internal layer has; the assumption $\pm p(\pm 1) < 0$ precludes layers at ± 1.

6.b) The variational formulation. The variational approach of [6] and [10] consists essentially of the following. The operator L_ε is selfadjoint with respect to the weighted inner product $(\cdot,\cdot)_w$,

$$(u,v)_w := \int_{-1}^{1} u(x)v(x)\exp\{-P(x)/\varepsilon\}dx , \tag{6.4}$$

$$P(x) := \int_{\beta_I}^{x} p(t)dt$$

where the index I is chosen such that $P(x) \leq 0$ for all $x \in [-1,1]$. Equivalent to problem (6.1) is the variational problem to find $u \in H_0^1(-1,1)$ which satisfies the boundary conditions $u(-1) = A$ and $u(1) = B$ and the variational form

$$B_\varepsilon(u,v) := \varepsilon(u',v')_w = 0, \quad \forall\, v \in H_0^1(-1,1) . \tag{6.5}$$

When we try to find approximations to the solution of (6.5), we see at once that the best candidates for approximations are among those ones which are almost constant everywhere,

except possibly near the minima $\beta_i (i = 0,\ldots,n)$ of the weight function, where internal transition layers may arise. In [6] and [10] a formal approximation is constructed, which is the solution of the restriction of B_ε to a finite dimensional subspace consisting of such functions. We shall prove here the validity of such a method by an argument common for Galerkin methods. For it we have to find a subspace which contains good approximations of the solution and of the test functions; moreover, we have to find a suitable positive lower bound for the bilinear form B_ε.

6.c) Construction of a formal approximation. In order to construct the subspace in which we look for an approximation of the solution u_ε of (6.1) we first construct error function-like formal approximations to the expected internal transition layer terms at $\beta_i (i = 0,\ldots,n)$ and, thereafter, we show that a linear combination thereof may yield a good approximation of u_ε. To this aim we define the functions $\rho_i, \varphi_i, \psi_i$ and χ_i:

ρ_i is a C^∞ cut-off function which satisfies

$$\rho_i(x) := \begin{cases} 1, & \text{if } |x - \beta_i| < \frac{1}{4}h, \\ 0, & \text{if } |x - \beta_i| > \frac{1}{2}h, \end{cases} \qquad h := \min_i \beta_i - \alpha_i ; \tag{6.6}$$

φ_i is an approximation to the non-constant part of P near β_i,

$$\varphi_i(x) := \sum_{j=2}^{2k-1} P^{(j)}(\beta_i)(x - \beta_i)^j/j! - \delta_i (x - \beta_i)^{2k} \tag{6.7}$$

in which δ_i is chosen such that $\varphi_i(x) \leq P(x) - P(\beta_i) - (x - \beta_i)^{2k}$, $\forall x$; ψ_i is an approximation of the error function-like transition layer at β_i,

$$\psi_i(x) := \mu_i(\varepsilon) \int_{-\infty}^{x} \rho_i(t) \exp\{\varphi_i(t)/\varepsilon\} dt , \tag{6.8}$$

in which $\mu_i(\varepsilon) = (-p'(\beta_i)/2\pi\varepsilon)^{\frac{1}{2}}(1 + 0(\varepsilon))$ is such that $\psi_i(x) = 0$ if $x \leq \alpha_i$ and $\psi_i(x) = 1$ if $x \geq \alpha_{i+1}$; χ_i is the function

$$\chi_i(x) := \psi_{i-1}(x) - \psi_i(x), \quad (i = 1,\ldots,n) \tag{6.9}$$

which is nearly the characteristic function of (β_{i-1},β_i) employed in [6] as test function in the variational form. The construction implies that $L_\varepsilon\psi_i$ is zero outside (α_{i-1},α_i) and that it satisfies the estimates

$$|L_\varepsilon\psi_i(x)| \le \mu_i C|x - \beta_i|^{2k-1}\exp\{\varphi_i(x)/\varepsilon\} \qquad (6.10a)$$

$$\le \varepsilon^{\frac{1}{2}-1/2k} C \exp\{(P(x) - P(\beta_i))/\varepsilon\}\ , \qquad (6.10b)$$

inside the interval.

A linear combination of these functions is to be an approximation of the solution of (6.1). This linear combination is determined by the discretization of the variational form (6.5). As test functions we take the subspace E,

$$E := \text{span}\{\chi_i,\ldots,\chi_n\}$$

and we seek a solution v_ε in the linear manifold $\psi_n + E$, which consists of all linear combinations of $\psi_i (i = 0,\ldots,n)$ that can satisfy the boundary conditions $u(-1) = 0$ and $u(1) = 1$,

$$v_\varepsilon := \sum_{i=0}^{n} \eta_i\psi_i, \quad \text{with} \quad \sum_{i=0}^{n} \eta_i = 1\ . \qquad (6.11)$$

This solution of the discretized variational form has to satisfy

$$B_\varepsilon(v_\varepsilon,\chi) = 0, \quad \forall\ \chi \in E\ . \qquad (6.12)$$

Taking a basis in E we find the set of n equations

$$0 = B_\varepsilon(v_\varepsilon,\chi_i) = \sum_{j=0}^{n} \eta_j(\psi_j',\psi_{i-1}' - \psi_i')_w =$$

$$= \eta_{i-1}\|\psi_{i-1}'\|_w^2 - \eta_i\|\psi_i'\|_w^2$$

From these equations and the side condition $\sum \eta_i = 1$, the coefficients can be solved; it is easily seen that they are all positive. Approximate evaluation of the integrals yields the equations

$$\eta_{i-1}\mu_{i-1}\exp(-P(\beta_{i-1})/\varepsilon) = \eta_i\mu_i\exp(-P(\beta_i)/\varepsilon)(1 + 0(\varepsilon))\ .$$

Since $P(\beta_i) \leq 0$ and $P(\beta_I) = 0$, this implies

$$\eta_i = 0(\exp(P(\beta_i)/\varepsilon)), \quad i = 0,\ldots,n \ . \tag{6.14}$$

6.d) A lower bound for the bilinear form. A lower bound for B_ε is derived with the aid of the maximum principle and a suitable barrier function. We shall employ the following generalization of [8, Theorems 14 and 17]:

Lemma 1: If a positive continuous function W exists, which is C^2 except in the points $\{x_1,\ldots,x_n\} \subset (-1,1)$ and which satisfies

$$L_\varepsilon W(x) \geq \Lambda W(x) \qquad \forall\, x \neq x_i \ (i = 1,\ldots,n) \ , \tag{6.15a}$$

$$W'(x_i - 0) \geq W'(x_i + 0) \qquad (i = 1,\ldots,n) \ , \tag{6.15b}$$

then the smallest eigenvalue of L_ε is not smaller than Λ. If moreover, $u \in C^2$ satisfies

$$u(\pm 1) = 0 \quad \& \quad |L_\varepsilon u(x)| \leq L_\varepsilon W(x), \qquad \forall\, x \neq x_i \ , \tag{6.16}$$

then u is bounded by W,

$$|u(x)| \leq W(x), \qquad \forall\, x \in [-1,1] \ .$$

Proof: Assume that $u - W$ has a positive maximum at an interior point $x = a \neq x_i (i = 1,\ldots,n)$, then $L_\varepsilon(u - W)$ is positive at a, which contradicts (6.16). If $u(x_i) - W(x_i)$ is a local maximum, then

$$u'(x_i) - W'(x_i - 0) > 0 \quad \text{and} \quad u'(x_i) - W'(x_i - 0) < 0 \ ;$$

this contradicts (6.15b). Hence $u - W$ has no interior maximum and is negative at the boundary, and thus negative everywhere. Likewise $-u - W$ is negative everywhere. For the assertion on the eigenvalue, we assume $L_\varepsilon u = \lambda u$ with $\lambda < \Lambda$; from $L_\varepsilon(u/w) < 0$ it then follows analogously that u/w has no interior maxima nor minima and hence that it is zero everywhere. □

A suitable barrier function is the function W,

$$W(x) = \min\left\{\int_x^1 (1+t)\exp\{P(t)/\varepsilon\}dt, \int_{-1}^x (1-t)\exp\{P(t)/\varepsilon\}dt\right\}; \tag{6.17}$$

it satisfies the correct jump condition across the discontinuity of W' and outside this point it satisfies

$$L_\varepsilon W(x) = \varepsilon \exp\{P(x)/\varepsilon\} .$$

Λ_ε is the minimum of this residue divided by W. Let ℓ be the index for which the expression

$$P(\alpha_i) - \min\{\max_{j<i} P(\beta_i), \max_{j>i} P(\beta_i)\} \tag{6.18}$$

takes its minimal value, and let m and r be the indices at which $\{P(\beta_i)\,|\,i < \ell\}$ and $\{P(\beta_i)\,|\,i > \ell\}$ respectively take their maximal value. Obviously, $r = I$, since P has an absolute maximum at β_I. Standard computations show:

$$\Lambda_\varepsilon = \min_x \varepsilon \exp(P(x)/\varepsilon)/W(x) = \tag{6.19}$$

$$= (-\varepsilon p'(\beta_m)/2\pi)^{\frac{1}{2}}\exp\{P(\alpha_\ell)/\varepsilon - P(\beta_m)/\varepsilon\}(1+0(\varepsilon)) .$$

It is easily seen from the minimax criterion (3.5) and the trial function $\psi_m - \psi_r$ that $C\varepsilon^{-\frac{1}{2}}\Lambda_\varepsilon$ is an upper estimate for the smallest eigenvalue. This shows that Λ_ε is a nearly optimal lower bound for B_ε.

6.e) Approximation properties of the subspace. The second point on which the proof of the validity of the variational approximation v_ε is based, is - as usual in Galerkin methods - that the approximate solution space $\psi + E$ contains a satisfactory approximation of the true solution u_ε. We shall show that such an approximation is given by U_ε,

$$U_\varepsilon := \psi_n + \sum_{i=1}^n u_\varepsilon(\alpha_i)\chi_i = \sum_{i=0}^n \zeta_i\psi_i , \tag{6.20}$$

where $\zeta_i := u_\varepsilon(\alpha_{i+1}) - u_\varepsilon(\alpha_i)$. Since it follows from the maximum principle that u_ε is monotonely increasing, all ζ_i are positive and smaller than or equal to one. Their size can be estimated much better.

Let us during this paragraph assume that P has only one absolute maximum in the interval $(-1,1)$, viz. β_I. Define the barrier function W_0,

$$W_0(x) := \begin{cases} \int_{-1}^{x} (\beta_I - \xi)\exp(P(\xi)/\varepsilon)d\xi/\varepsilon, & \text{if } x \leq \beta_I , \\ \gamma \int_{x}^{1} (\xi - \beta_I)\exp(P(\xi)/\varepsilon)d\xi/\varepsilon, & \text{if } x \geq \beta_I , \end{cases} \tag{6.21}$$

where $\gamma = (1 + 0(\varepsilon))$ is chosen such that W_0 is continuous at β_I. As in (6.17) this function satisfies

$$L_\varepsilon W_0 = \exp\{P(x)/\varepsilon\} ; \tag{6.22}$$

moreover, its derivative is continuous at β_I and it has a maximum there of order unity. More generally we have

$$W_p(\alpha_i) = 0(\max_{j<i} \varepsilon^{-\frac{1}{2}}\exp(P(\beta_j)/\varepsilon)), \quad \text{if } i \leq I , \tag{6.23}$$

and an analogous formula, where maximum is taken over $j \geq i$, if $i \geq I$. Let now Y be a linear combination of ψ_i,

$$Y = \sum_{i=0}^{n} \nu_i\psi_i, \qquad \sum_{i=0}^{n} \nu_i = 1 , \tag{6.24a}$$

in which the orders of the coefficients are prescribed by the conditions

$$\nu_i = 0(\exp(P(\beta_i)/\varepsilon)), \qquad i = 0,\ldots,n . \tag{6.24b}$$

According to formula (6.10b), this linear combination satisfies the estimate

$$L_\varepsilon Y \leq C\varepsilon^{\frac{1}{2}-1/2k}\exp\{P(x)/\varepsilon\} .$$

Since $u_\varepsilon - Y$ has zero boundary values, we can apply the second half of Lemma 1 with barrier function $\varepsilon^{\frac{1}{2}-1/2k}W_0$. This implies that $u_\varepsilon - Y$ is bounded by $\varepsilon^{-1/2k}W_0$, and in particular this shows

$$\zeta_i = u_\varepsilon(\alpha_{i+1}) - u_\varepsilon(\alpha_i) = 0(\varepsilon^{-1/2k}W_0(\alpha_i)) .$$

At at least one point α_i at the left-hand side of β_I we thus obtain an estimate for ζ_i of the form

$$\zeta_i = 0(\varepsilon^{-1/2k}\exp(P(\beta_i)) \ . \qquad (6.26$$

and analogously at the right-hand side of β_I.

This process sketched above we can apply to each subinterval (α_i,α_j) of $(-1,1)$, which contains only one absolute maximum of P. Hence, starting with intervals containing the largest maxima, we can go down step by step, in each step establishing an estimate for at least two coefficients ζ_i and loosing in each step a factor of order $\varepsilon^{-1/2k}$. Starting with a sufficiently large k, we obtain in this way the estimate

$$\zeta_i = 0(\varepsilon^{-\frac{1}{2}}\exp(P(\beta_i)), \quad i = 0,\ldots,n \ . \qquad (6.27)$$

For an estimate of the global error of the approximation U_ε, which we have constructed in $\psi_n + E$, we consider a subinterval (α_i,α_{i+1}). At both endpoints $U_\varepsilon - u_\varepsilon$ is zero by definition; moreover, the estimates (6.10a) and (6.27) imply

$$\|L_\varepsilon(U_\varepsilon - u_\varepsilon)\|_{w,i} \leq C\varepsilon^{k-2} \ , \qquad (6.28a)$$

where $\|\cdot\|_{w,i}$ is the restriction of the norm to the subinterval (α_i,α_{i+1}). Since this subinterval does not contain in its interior any zero of p at which p' is positive, the smallest eigenvalue of the restriction of L_ε to this subinterval (with Dirichlet boundary conditions) is bounded away from zero. Hence (6.28a) implies that $(U_\varepsilon - u_\varepsilon)$ is of the same order. Adding up over all subintervals, we thus find:

$$\|U_\varepsilon - u_\varepsilon\|_w \leq C\varepsilon^{k-2} \ . \qquad (6.28b)$$

6.f) Error bounds for the variational approximation. Having now a lower bound for B_ε and a good approximation in $\psi + E$, we can apply the usual Galerkin-argument. From (6.5) and (6.12) we find

$$B_\varepsilon(u_\varepsilon - v_\varepsilon,\chi) = 0, \quad \forall\ \chi \in E \ .$$

Adding and subtracting U_ε we find

$$B_\varepsilon(U_\varepsilon - v_\varepsilon, \chi) = B_\varepsilon(U_\varepsilon - u_\varepsilon, \chi) \ .$$

Since $U_\varepsilon - v_\varepsilon$ is an element of E, we may choose χ equal to this. Hence we find

$$B_\varepsilon(U_\varepsilon - v_\varepsilon, U_\varepsilon - v_\varepsilon) = B_\varepsilon(U_\varepsilon - u_\varepsilon, U_\varepsilon - v_\varepsilon) =$$
$$= 2B_\varepsilon(U_\varepsilon - u_\varepsilon, U_\varepsilon - v_\varepsilon) - B_\varepsilon(U_\varepsilon - v_\varepsilon, U_\varepsilon - v_\varepsilon) \leq$$
$$\leq B_\varepsilon(U_\varepsilon - u_\varepsilon, U_\varepsilon - u_\varepsilon) \ ,$$

where we used the estimate

$$B_\varepsilon(u,v) \leq (B_\varepsilon(u,u)B_\varepsilon(v,v))^{\frac{1}{2}} \leq \frac{1}{2}B_\varepsilon(u,u) + \frac{1}{2}B_\varepsilon(v,v) \ .$$

Together with the lower bound (6.19) this implies

$$\|U_\varepsilon - v\|_w^2 \leq C\varepsilon^{2k-9/2}\exp\{\frac{1}{\varepsilon}\,P(\beta_m) - \frac{1}{\varepsilon}\,P(\alpha_\ell)\} \ .$$

This does not at all look like a satisfactory estimate. However, if we evaluate $\|U_\varepsilon - v_\varepsilon\|_w$ approximately, we find

$$\|U_\varepsilon - v_\varepsilon\|_w^2 =$$
$$= \sum_{i=1}^{n} \sum_{j=0}^{i-1} (\zeta_j - \eta_j)^2 (2\pi\varepsilon/p'(\alpha_i))^{\frac{1}{2}} \exp\{-P(\alpha_i)/\varepsilon\}(1 + 0(\varepsilon)),$$

hence at least at α_ℓ we find a good estimate, namely

$$|U_\varepsilon(\alpha_\ell) - v_\varepsilon(\alpha_\ell)|^2 = \sum_{j=0}^{\ell-1} (\zeta_j - \eta_j)^2 \leq$$
$$\leq C\varepsilon^{2k-9/2}\exp\{P(\beta_m)/\varepsilon\} \ .$$

This estimate we get precisely at the point in the interval, that "generates" the smallest eigenvalue of L_ε. If there are several of such points, i.e. if (6.18) does not determine a unique ℓ, then at all those points such an estimate holds.

Now we can split the interval in (at least) two subintervals, $(-1,\alpha_\ell)$ and $(\alpha_\ell,1)$ and restrict the problem (6.1) to both subintervals. Since α_ℓ is now a boundary point for both problems, it does not generate an eigenvalue that tends to zero, see Theorem 1 and the comments that

follow it. Hence, applying the same proof as before, we get error estimates at two other zeros of p, which in the restricted problems generate the smallest eigenvalues. So we can go on until we have obtained error estimates of type (6.29) for all unknowns in the variational approximation. Thereafter we can easily derive error estimates for all other points of the interval by the maximum principle. So we have finally proved:

Theorem 3: The approximation v_ε, generated by the discretized variational form (6.12) satisfies the error estimate

$$|u_\varepsilon(x) - v_\varepsilon(x)| \leq C\varepsilon^{k-9/4} . \tag{6.30}$$

Remarks:

i) If $p(-1) < 0$, we have to add to the set of trial functions $\{\psi_i\}$ a trial function whose exponential part is of the form $\exp(p(-1)(x+1)/\varepsilon)$ and which represents an ordinary boundary layer. If $p(1) > 0$ we do analogously.

ii) If $p(-1) = 0$ and $p'(-1) < 0$, we have to add to the set of trial functions a function of the same type as the other ψ_i, whose jump is concentrated near -1. If $p(1) = 0$ and $p'(1) = 0$ we do the same at $x = +1$.

iii) If p has a multiple zero, this zero generates a point-spectrum that becomes dense on the whole positive real axis, if ε tends to zero, cf. [11]. More specifically, if $p^{(k)}(\alpha) \neq 0$ and $p(x) = 0((x-\alpha)^k)$ $(x \to \alpha)$, then the distance between two subsequent eigenvalues is of the order $0(\varepsilon^\gamma), \gamma := (k-1)/(k+1)$. If p changes sign at α with a non-negative slope, the smallest eigenvalue generated at this point is of exponentially small order and it has to be taken in account as before. If p changes sign with non-positive slope, or if p does not change sign, the smallest eigenvalue is bounded away from zero by a distance of order $0(\varepsilon^\gamma)$, hence we loose only an extra factor $\varepsilon^{-\gamma}$ in going from (6.28a) to (6.28b). However, at a point α where p changes sign a boundary layer function has to be constructed and added to the trial space; the boundary layer is of width $\varepsilon^{1/(k+1)}$.

iv) The whole analysis carries over to problems governed by equations of type $-\varepsilon\Delta u + \underline{p}^T\nabla u = 0$ on a bounded domain in several dimensions, cf. [6], [10] and [12].

v) The estimate (6.19) yields a better estimate of the first eigenvalue of L_ε than the formulae of Friedman and Ventcel and Freidlin, cf. [13]. It looks even not too difficult to obtain an approximation with a relative error of order $0(\varepsilon)$ by considering the approximation of its eigenfunction in E in somewhat more detail.

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We study the singularly perturbed two-point boundary value problem on the interval $(-1,1)$

$$-\varepsilon u'' + p(x)u' + q(x)u = 0, \quad u(-1) = A, \quad u(1) = B,$$

in which the coefficient p has zeros in the interval $(-1,1)$. For the case

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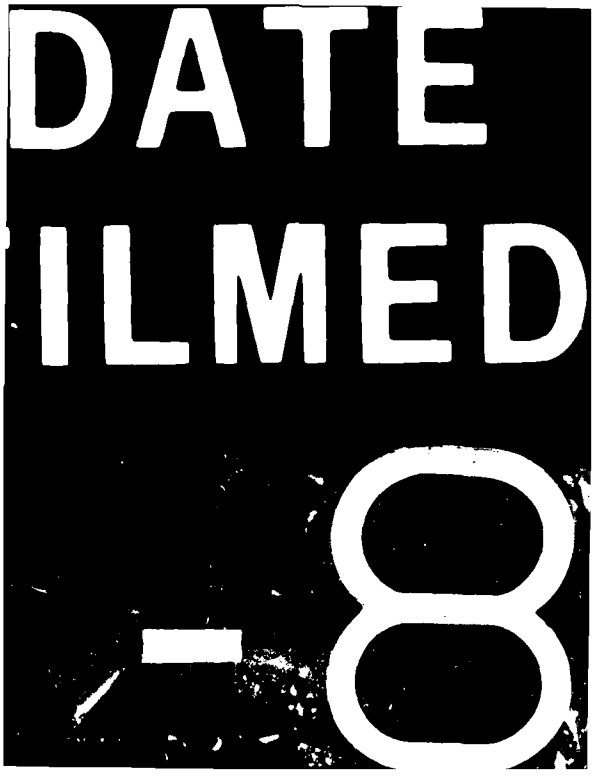
20. ABSTRACT - Cont'd.

where p has precisely one zero we give a survey of our method of construction of an asymptotic approximation of the solution via the eigenfunction expansion, and we show that "Ackerberg - O'Malley resonance" is identical to ordinary resonance, namely that a free mode in the solution is amplified strongly by a small divisor. For the case where p has several zeros and where q is identically zero we construct an asymptotic approximation and prove its validity by a variational (Galerkin) method. The methods described in this paper can be generalized to turning point problems in several dimensions.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